Design of Mechanisms

Rigid Body Guidance and Function Generation

Michael M. Stanišić

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UNIVERSITY OF NOTRE DAME

Outline

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   - Circling & Center Points
   - Example
2. Function Generation
   - Prob. Def.
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   - Example

Problem Definition

Rigid Body Guidance - Design a mechanism so that a rigid body is guided through the three positions, A_1B_1, A_2B_2 and A_3B_3.

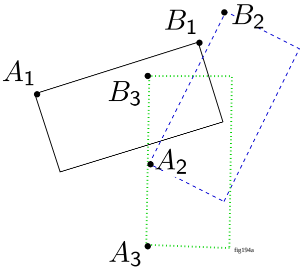


Figure: Three desired positions of a rigid body AB

This is the Three Position Problem. There are also Four and Five Position Problems.

Circling & Center Points

A is a circling point and O_a is the respective center point.

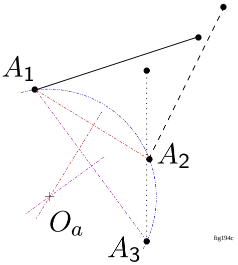


Figure: The circle containing A_1, A_2 and A_3

B is a circling point and O_b is the respective center point.

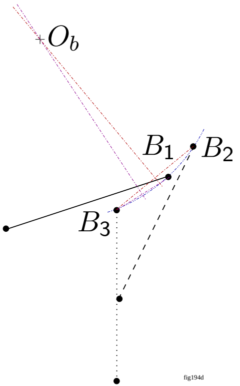


Figure: The circle containing B_1, B_2 and B_3

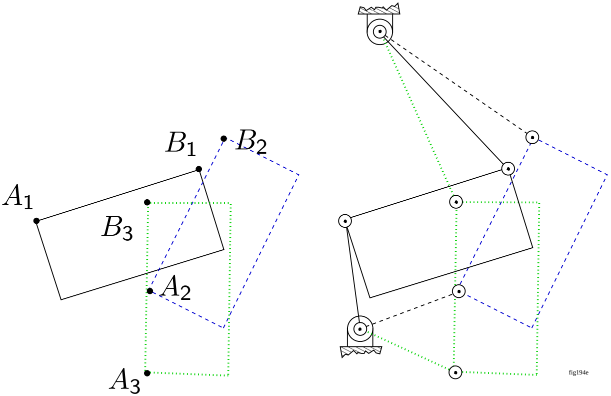
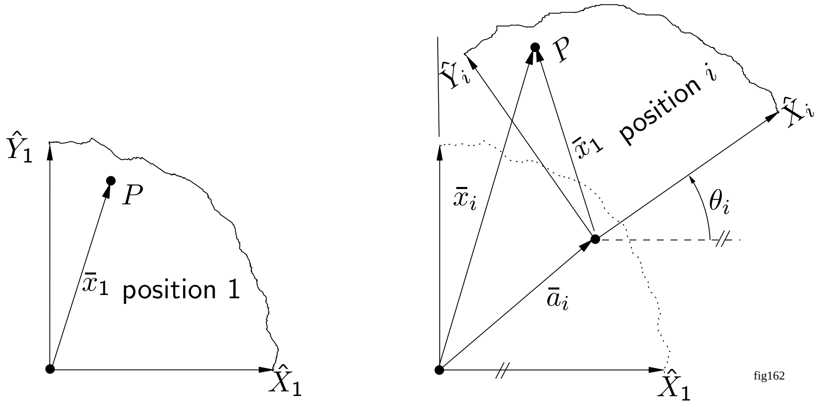


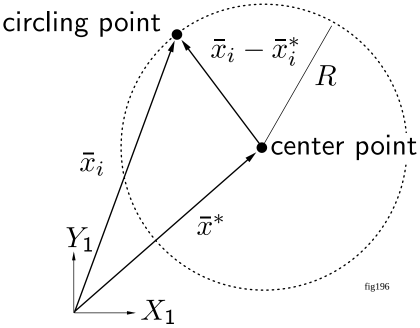
Figure: One of an ∞^4 of Solutions

Model of a Rigid Body Displacement



$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} a_{ix} \\ a_{iy} \end{bmatrix}$$

$\bar{x}_i = [x_i, y_i]^T$ are coordinates of a circling point,
$\bar{x}^* = [x^*, y^*]^T$ are coordinates of corresponding center point
R is the radius of the circle



$$(\bar{x}_1 - \bar{x}^*) \cdot (\bar{x}_1 - \bar{x}^*) = (\bar{x}_i - \bar{x}^*) \cdot (\bar{x}_i - \bar{x}^*) \quad (i = 2, 3)$$

2 scalar equations in 4 scalar unknowns, $\bar{x}_1$ and $\bar{x}^*$.

Given the circling point $\bar{x}_1$, find the center point $\bar{x}^*$

$$x^* = \frac{\begin{vmatrix} W_2 & V_2 \\ W_3 & V_3 \end{vmatrix}}{\begin{vmatrix} U_2 & V_2 \\ U_3 & V_3 \end{vmatrix}} \quad \text{and} \quad y^* = \frac{\begin{vmatrix} U_2 & W_2 \\ U_3 & W_3 \end{vmatrix}}{\begin{vmatrix} U_2 & V_2 \\ U_3 & V_3 \end{vmatrix}}$$

where,

$$U_i = x_1(\cos\theta_i - 1) - y_1 \sin\theta_i + a_{ix}$$
$$V_i = x_1 \sin\theta_i + y_1(\cos\theta_i - 1) + a_{iy}$$
$$W_i = \frac{a_{ix}^2 + a_{iy}^2}{2} + x_1(a_{ix}\cos\theta_i + a_{iy}\sin\theta_i) + y_1(a_{iy}\cos\theta_i - a_{ix}\sin\theta_i)$$

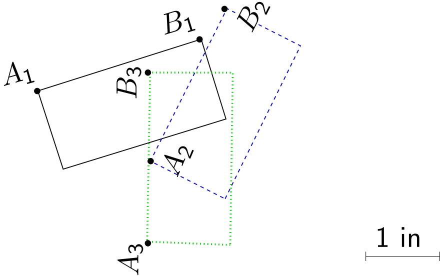
Given the center point $\bar{x}^*$, find the center point $\bar{x}_1$

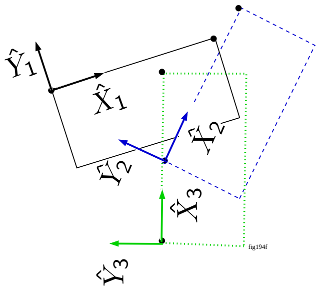
$$x_1 = \frac{\begin{vmatrix} C_2 & B_2 \\ C_3 & B_3 \end{vmatrix}}{\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}} \quad \text{and} \quad y_1 = \frac{\begin{vmatrix} A_2 & C_2 \\ A_3 & C_3 \end{vmatrix}}{\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}}$$

where,

$$A_i = x^*(\cos\theta_i - 1) + y^* \sin\theta_i - a_{ix}\cos\theta_i - a_{iy}\sin\theta_i$$
$$B_i = -x^*\sin\theta_i + y^*(\cos\theta_i - 1) + a_{ix}\sin\theta_i - a_{iy}\cos\theta_i$$
$$C_i = \frac{a_{ix}^2 + a_{iy}^2}{2} - a_{ix}x^* - a_{iy}y^*$$

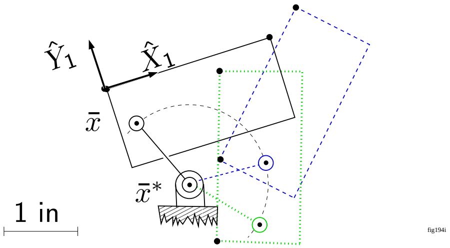
Example





$$\text{Position 2}: \quad \bar{a}_2 = \begin{bmatrix} 1.1842 \\ -1.4050 \end{bmatrix} \text{inches} \quad , \quad \theta_2 = 46.36^\circ$$

$$\text{Position 3}: \quad \bar{a}_3 = \begin{bmatrix} 0.7917 \\ -2.4392 \end{bmatrix} \text{inches} \quad , \quad \theta_3 = 71.90^\circ$$

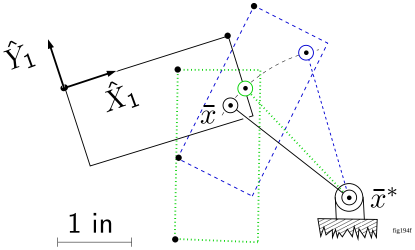


Take:

$$\bar{x} = \begin{bmatrix} 0.246 \\ -0.573 \end{bmatrix} \text{inches}$$

Compute:

$$\bar{x}^* = \begin{bmatrix} 0.677 \\ -1.58 \end{bmatrix} \text{inches}$$

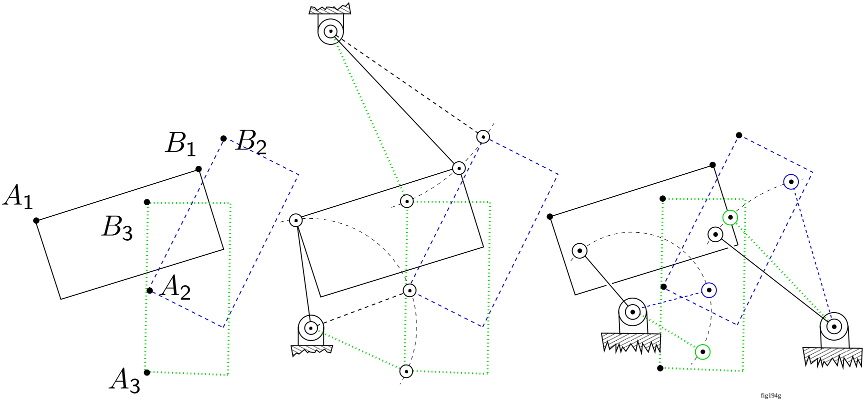


Take:

$$\bar{x} = \begin{bmatrix} 2.06 \\ -0.912 \end{bmatrix} \text{inches}$$

Compute:

$$\bar{x}^* = \begin{bmatrix} 3.22 \\ -2.58 \end{bmatrix} \text{inches}$$

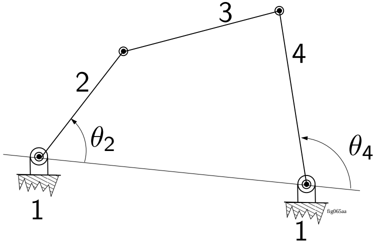


B_1
B_2
A_1
B_3
A_2
A_3
fig194g

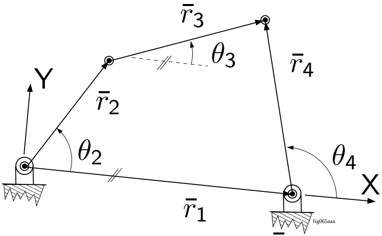
Problem Definition

Function Generation - Design a mechanism so that the output rotation θ_4 is a desired function of the input rotation θ_2.

$$\theta_4 = f(\theta_2)$$



The Vector Loop



$$\bar{r}_2 + \bar{r}_3 - \bar{r}_4 - \bar{r}_1 = \bar{0}$$

$$r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_4 \cos\theta_4 - r_1 = 0$$
$$r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_4 \sin\theta_4 = 0$$

$$r_3 \cos\theta_3 = -r_2 \cos\theta_2 + r_4 \cos\theta_4 + r_1$$
$$r_3 \sin\theta_3 = -r_2 \sin\theta_2 + r_4 \sin\theta_4$$

$$r_3^2 = r_2^2 + r_4^2 + r_1^2 - 2r_2 r_4 \cos\theta_2 \cos\theta_4$$
$$+2r_1 r_4 \cos\theta_4 - 2r_2 r_1 \cos\theta_2 - 2r_2 r_4 \sin\theta_2 \sin\theta_4$$

$$r_3^2 = r_2^2 + r_4^2 + r_1^2 - 2r_2r_4\cos\theta_2\cos\theta_4$$
$$+2r_1r_4\cos\theta_4 - 2r_2r_1\cos\theta_2 - 2r_2r_4\sin\theta_2\sin\theta_4$$

$$r_2^2 - r_3^2 + r_4^2 + r_1^2 + 2r_1r_4\cos\theta_4 - 2r_2r_1\cos\theta_2 =$$
$$2r_2r_4\cos\theta_2\cos\theta_4 + 2r_2r_4\sin\theta_2\sin\theta_4$$

dividing through both sides by $2r_2r_4$

$$\underbrace{\frac{r_2^2 - r_3^2 + r_4^2 + r_1^2}{2r_2r_4}}_{R_3} + \underbrace{\frac{r_1}{r_2}}_{R_2}\cos\theta_4 - \underbrace{\frac{r_1}{r_4}}_{R_1}\cos\theta_2 =$$

$$\underbrace{\cos\theta_2\cos\theta_4 + \sin\theta_2\sin\theta_4}_{\cos(\theta_2-\theta_4)}$$

Freudenstein's Equation

$$R_3 + R_2 \cos\theta_4 - R_1 \cos\theta_2 = \cos(\theta_2 - \theta_4)$$

where:

$$R_1 = \frac{r_1}{r_4} \;\; , \;\; R_2 = \frac{r_1}{r_2} \;\text{ and }\; R_3 = \frac{r_2^2 - r_3^2 + r_4^2 + r_1^2}{2 r_2 r_4}$$

Given three pairs of θ_2 and θ_4 values, Freudenstein's Eqn can be written three times giving a system of three equations which are linear in the three unknowns R_1, R_2 and R_3.

Example

Design a four bar linkage to replace a pair of gears with a 2:1 ratio, i.e. a four bar mechanism that generates the function, $\theta_2 = 2\theta_4$.

θ_2	θ_4	$\cos\theta_4$	$\cos\theta_2$	$\cos(\theta_2 - \theta_4)$
70°	10°	0.985	0.342	0.500
100°	25°	0.906	-0.174	0.259
130°	40°	0.766	-0.643	0.000

$$R_3 + 0.985R_2 - 0.342R_1 = 0.500$$
$$R_3 + 0.906R_2 + 0.174R_1 = 0.259$$
$$R_3 + 0.766R_2 + 0.643R_1 = 0.000$$

Solution,

$$R_3 = -0.2061 \quad , \quad R_2 = 0.5858 \quad , \quad R_1 = -0.3774$$

$$R_3 = -0.2061 \quad , \quad R_2 = 0.5858 \quad , \quad R_1 = -0.3774$$

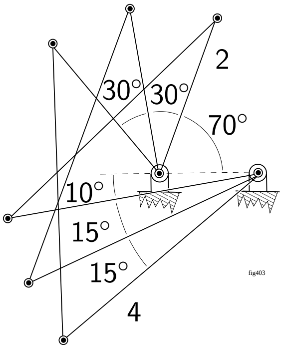
Take $r_1 = 10$ inches, and solve

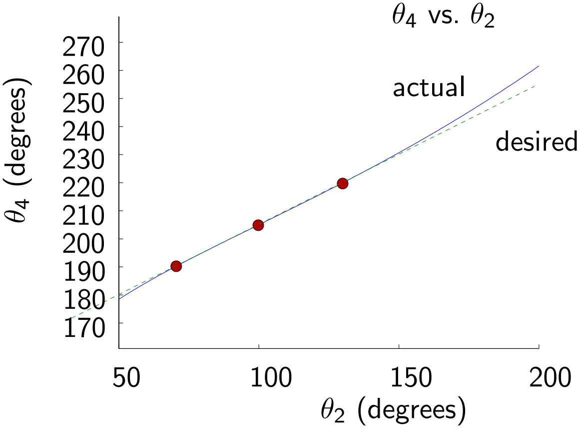
$$R_1 = \frac{r_1}{r_4} \ , \ R_2 = \frac{r_1}{r_2} \text{ and } R_3 = \frac{r_2^2 - r_3^2 + r_4^2 + r_1^2}{2r_2r_4}$$

for r_2, r_3 and r_4,

$r_2 = 17.071$ inches , $r_4 = -26.497$ inches , $r_3 = 30.117$ inches

θ_2	θ_4
70°	10°
100°	25°
130°	40°




θ4 vs. θ2
270
260
250
240
230
220
210
200
190
180
170
θ4 (degrees)
actual
desired
50
100
150
200
θ2 (degrees)